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AXISYMMETRIC BUCKLING AND GROWTH OF A CIRCULAR DELAMINATION IN A COMPRESSED LAMINATE

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Abstract—The energy-release rate associated with uniform-expansion growth of a circular delamination in a compressively loaded laminate is obtained by means of the M-integral. An iterative procedure based on the fourth-order Runge-Kutta integration formula is used to generate a family of nondimensionalized postbuckling solutions of von Karman's nonlinear plate theory. These solutions determine the change of the energy-release rate with the growth of a circular thin-film or midplane delamination, which in turn determines the stability characteristics of growth.

1. INTRODUCTION

In a previous article[1], a simple analytical expression of the energy-release rate was obtained for the postbuckling growth of a one-dimensional delamination in a compressively loaded orthotropic laminate. To a degree of approximation consistent with the classical beam-plate theory, it was found that the energy-release rate is a function of the axial forces and the bending moments acting upon the various cross-sections adjoining the moving front of delamination. Consequently, if the postbuckling solution at a certain stage of delamination growth has been obtained, the energy-release rate at that state is immediately determinable, and on the basis of the Griffith fracture criterion, the viability of continued growth of delamination can be inferred.

In this article, I study the parallel problem of axisymmetric buckling and uniform-expansion growth of a circular delamination in a compressively loaded plate. As an analytical model for investigating the residual strength of a laminated structure with impact-generated damage, a circular delamination is obviously more appropriate than a one-dimensional one. The critical load for the growth of a thin circular delamination in an axisymmetrically compressed plate has been computed by Kachanov[2] on the basis of linear postbuckling equations. However, a proper formulation of the postbuckling problem of a circular delamination requires von Karman's geometrically nonlinear plate theory, in which the transverse deflection, membrane compression and bending moment of the delaminated layer are to be determined from a system of two second-order nonlinear ordinary differential equations. The complexity of this system of equations precludes its solution in closed analytical form, and requires a numerical iterative procedure for finding solutions that satisfy the boundary conditions at the center and the edge of delamination. The development of the iterative procedure is considerably simplified by observing that at the boundary of delamination, the radial displacement of the postbuckling solution depends linearly on the radial compression and its first derivative [see eqn (5) of the present article]. This interesting and useful relation is derived from a consideration of the curve length of the profile of the buckled delamination.

By implementing the iterative procedure on the basis of nondimensionalized governing equations, a one-parameter family of nondimensionalized postbuckling solutions can be generated. Each solution of the family corresponds to a unique value of the yet undetermined delamination radius, and the association of the delamination radius to the postbuckling solution is determined by the compressive edge loading upon the laminate. Furthermore, at each stage of uniform-expansion growth of the delamination, the strain energy-release rate may be evaluated by means of the path-independent M-integral.† The resulting expression, eqn (12), is given in terms of the discontinuities in the axial force and the bending moment across the boundary of the delaminated layer. Consequently, under a specified edge loading on the laminate, each nondimensionalized postbuckling solution in the one-parameter family corresponds to a unique pair of values of the delamination radius and the energy-release rate. The change of the energy-release rate with the increase of delamination radius determines the stability characteristics of delamination growth under the specified laminate load.

In the present paper, the laminate is assumed to be a homogeneous isotropic elastic plate with elastic moduli E and ν. Only two types of delamination models are considered: the "thinfilm model," whose buckling deformation produces no effects on the stress and deformation in the main body of the laminate, and the symmetric "two-layer model," in which a circular delamination exists and grows in the midplane of the laminate. Under a specified isotropic compression in the laminate, the energy-release rate is found to increase monotonically with the growth of a circular thin-film delamination. Therefore, in contrast to the variegated growth behavior of one-dimensional delaminations[3], the uniform-expansion growth of a circular thin-film delamination, once started, is always catastrophic.

For a symmetric two-layer model under fixed compressive force at the boundary of the laminate, it is found that the energy-release rate also increases monotonically with the growth of delamination. Furthermore, the increase of the energy-release rate accelerates as the delamination front approaches the boundary of the laminate. This result implies the catastrophic nature of growth, and is in contradiction to the recent conclusion of Bottega and Maewal[4; cf. 5]. On the other hand, if the boundary of the laminate is subjected to a fixed inward displacement, then in general the energy-release rate at first increases with delamination growth and subsequently decreases. The larger the specified inward displacement at the boundary, the smaller is the delamination radius at which the energy-release rate reaches its peak value.

2. POSTBUCKLING PROBLEM OF A CLAMPED CIRCULAR PLATE

The critical buckling load of a two-dimensional delamination can be obtained by applying the linear buckling equations of the classical plate theory. For a clamped isotropic circular plate under axisymmetric in-plane compression, the transverse deflection as a function of the radial coordinate satisfies Bessel's equation of the zeroth order, and the critical radial compressive force P_{cr} per unit arc length of the boundary of the delamination is determined by the first zero of Bessel's function of the first order[6]:

$$J_1\left[a\left(\frac{P_{cr}}{D}\right)^{1/2}\right] = 0 \tag{1a}$$

or

$$a\left(\frac{P_{cr}}{D}\right)^{1/2} = 3.8317, \tag{1b}$$

where $D = Eh^3/[12(1 - \nu^2)]$ and a and h denote, respectively, the radius and thickness of the delamination.

In general, a reasonably accurate postbuckling analysis of the delaminated layer requires the application of von Karman's geometrically nonlinear plate theory. In contrast to the linear theory, the membrane stress in the nonlinear buckling theory is neither transversely isotropic nor uniform over the plate. For the case of a clamped circular

† The strain energy-release rate is the incremental strain energy release per unit gain of delamination area.

plate under axisymmetric compression, the radial compressive force per unit circumferential length depends on the radial coordinate, $P = P(r)$. Let

$$\phi(r) = \frac{w'(r)}{r}, \tag{2}$$

where $w = w(r)$ is the transverse displacement. Then the postbuckling problem is described by the following system of differential equations and boundary conditions[7]:

$$r^{-3}(r^3P')' = \frac{Eh\phi^2}{2}, \qquad r^{-3}(r^3\phi')' = -\frac{P\phi}{D}, \tag{3a, b}$$

$$P'(0) = 0, \qquad \phi'(0) = 0, \qquad \phi(a) = 0, \tag{4a, b, c}$$

where the primes indicate differentiation with respect to r. If the compressive loading on the boundary ($r = a$) produces a radial displacement $u(a) = -\delta$, then

$$\begin{aligned}
\delta &= -\int_0^a \epsilon_r\, dr + \int_0^a \frac{(w')^2}{2}\, dr \\
&= -\left(\frac{1}{E}\right)\int_0^a (\sigma_r - \nu\sigma_\theta)\, dr + \int_0^a \frac{r^2\phi^2}{2}\, dr \\
&= -\left(\frac{1}{Eh}\right)\int_0^a (-P + \nu(rP)')\, dr + \left(\frac{1}{Eh}\right)\int_0^a \frac{1}{r}(r^3P')'\, dr \\
&= \left(\frac{1}{Eh}\right)[(1 - \nu)aP(a) + a^2P'(a)],
\end{aligned} \tag{5}$$

where eqn (3a) has been used.

A boundary-value problem is defined by eqns (3) and (4) together with a specified edge load $P(a)$. In the case of a specified edge displacement $-\delta$, the boundary-value problem is defined by eqns (3)–(5). Solution of the boundary-value problem yields the radial force $P(a)$ and the bending moment $M_r(a)$ per unit arc length of the boundary, where

$$M_r = D(w'' + \nu w'/r) = D[(r\varphi)' + \nu\phi]. \tag{6}$$

In the following section, the energy-release rate associated with uniform-expansion growth of a circular thin-film delamination is obtained in terms of $P(a)$, $M_r(a)$ and the compressive strain in the laminate.

3. THE M-INTEGRAL AND THE ENERGY RELEASE RATE

If a circular thin-film delamination buckles when the laminate is subjected to transversely isotropic compressive strain ϵ_0, the radial displacement at the boundary of the delamination is $u(a) = -\delta = -a\epsilon_0$. The radial compressive force $P(a)$ and bending moment $M_r(a)$ can be computed by solving eqns (3)–(5) numerically. In the following analysis, the path-independent M-integral is used to obtain the strain energy-release rate in terms of ϵ_0, $P(a)$ and $M_r(a)$.

Let a constant biaxial stress field

$$\sigma_x = \sigma_y = \frac{\epsilon_0 E}{1 - \nu}, \qquad \sigma_z = 0, \qquad \tau_{ij} = 0 \tag{7}$$

be superposed on the postbuckling solution. Since the constant stress field is continuous, it produces no effect on the stress-intensity factors and the energy-release rate. With the superposition of the constant stress field, the region of the laminate sufficiently far away from the delamination becomes stressfree while the delaminated layer is subjected to a bending moment M^* and a *tensile* radial force P^* per unit arc length of the boundary given, respectively, by

$$M^* = M_r(a) = D[(r\phi)' + \nu\phi]\,|_{r=a}, \qquad P^* = \frac{Eh\epsilon_0}{1-\nu} - P(a). \tag{8a, b}$$

Let $W = \sigma_{ij}\epsilon_{ij}/2$ be the strain-energy density. Then the following expressions are valid along the edge of the delamination:

$$\sigma_r = \frac{P^*}{h} - \frac{12M^*\eta}{h^3}, \qquad \frac{-h}{2} \leqq \eta \leqq \frac{h}{2} \tag{9a}$$

$$E\epsilon_\theta = \sigma_\theta - \nu\sigma_r = 0, \qquad \epsilon_r = (1-\nu^2)\frac{\sigma_r}{E} \tag{9b, c}$$

$$Wx_jn_j = -(1-\nu^2)a\frac{\sigma_r^2}{2E}, \qquad n_j = -\frac{x_j}{r} \tag{9d, e}$$

$$-\sigma_{jk}n_ku_{j,i}x_i = (1-\nu^2)\frac{a\sigma_r^2}{E}. \tag{9f}$$

Budiansky and Rice[8] have shown that the energy-release rate in uniform-expansion growth of a crack is related to the path-independent M-integral:

$$a\frac{d\Pi}{da} = M = \iint Wx_in_i - \sigma_{jk}n_ku_{j,i}x_i - \sigma_{ik}n_k\frac{u_i}{2}\,dA, \tag{10}$$

where Π is the released energy and the integral on the right-hand side extends over an arbitrary surface enclosing the circular boundary of the delamination. An admissible choice of the surface of integration is a surface of revolution generated by the boundary curve of the hatched region in Fig. 1. Over that portion of the integration surface that intersects the delaminated layer, the contribution to the M-integral can be calculated

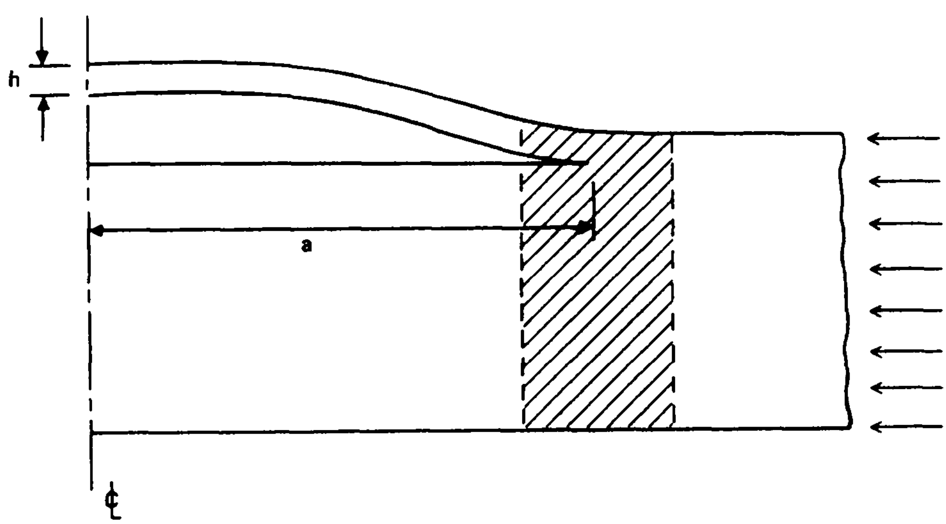

Fig. 1. Circular thin-film delamination.

on the basis of eqn (9). The result is

$$a\frac{d\Pi}{da} = 2\pi a^2 \int_{-h/2}^{h/2} \frac{1}{2E}(1 - \nu^2)\sigma_r^2 \, d\eta$$
$$= \frac{\pi a^2(1 - \nu^2)}{Eh}\left[(P^*)^2 + 12\left(\frac{M^*}{h}\right)^2\right]. \quad (11)$$

The portion of the integration surface that intersects the laminate underneath the delaminated layer yields a contribution of the order $h/(t - h)$ compared with the right-hand side of eqn (11), where t is the total thickness of the laminate. The remaining portions of the integration surface make no contribution to the M-integral.† In the limit of thin-film delamination, $h/t \to 0$, eqns (8) and (11), deliver the strain energy-release rate per unit increment of the area of delamination:

$$G = \frac{d\Pi}{d\pi a^2} = \frac{M}{2\pi a^2}$$
$$= (1 - \nu^2)\frac{(P^*)^2 + 12(M^*/h)^2}{2Eh}$$
$$= \frac{1 - \nu^2}{2Eh}\left\{\left[\frac{Eh}{1 - \nu}\epsilon_0 - P(a)\right]^2 + 12\left[\frac{M_r(a)}{h}\right]^2\right\}. \quad (12)$$

In the preceding derivation of the M-integral, the singular stress field of the elasticity solution in the vicinity of the crack front has been replaced by a nonsingular distribution of membrane and bending stresses across the thickness of the delaminated layer, as expressed by eqn (9a). This substitution reduces the M-integral of eqn (10) (defined in terms of the stresses and displacements of the elasticity solution) to the simple expression of eqn (11). Its legitimacy can be justified on the ground that the M-integral is independent of the surface of integration. Thus, the surface $\mathcal{S}_0$ generated by revolution of the hatched region in Fig. 1 can be replaced by another surface $\mathcal{S}$ enclosing the delamination front yet sufficiently far away from it so that the stresses and displacements over the new surface $\mathcal{S}$ are accurately predicted by the Kirchhoff-Love assumption, i.e.

$$\iint_{\mathcal{S}_0} dM = \iint_{\mathcal{S}} dM \approx \iint_{\mathcal{S}} dM',$$

where the integral $\iint dM'$ is based on the stresses and displacements of the plate theory. Now the integral $\iint dM'$ vanishes over any closed surface enclosing a region in which the integrand has no singularity. Hence, $\iint_{\mathcal{S}} dM'$ does not change its value when any one of the three cylindrical surfaces of $\mathcal{S}$ is replaced by a corresponding cylindrical surface of $\mathcal{S}_0$. It follows that

$$\iint_{\mathcal{S}} dM' = \iint_{\mathcal{S}_0} dM'.$$

The last integral delivers the result of eqn (12).

4. CATASTROPHIC GROWTH OF A CIRCULAR THIN-FILM DELAMINATION

Under a specified compressive strain in the laminate $\epsilon_x = \epsilon_y = -\epsilon_0$, a postbuckling solution of the delaminated layer can be obtained from eqns (3)–(5), where $\delta = a\epsilon_0$.

† It should be recalled that the constant stress field (7) has been superimposed upon the postbuckling solution so that the region of the laminate sufficiently far away from the delamination is free from stress.

The postbuckling solution changes as the delamination radius increases. At each stage of growth, the energy-release rate is given by eqn (12) in terms of the boundary values of the postbuckling solution. The change of the energy-release rate G with the increase of the delamination radius determines the stability of delamination growth. In this section, it is shown that the dependence of G upon the delamination radius can be efficiently determined by an iterative procedure based on the shooting method. The system of eqns (3)–(5) is nondimensionalized and treated as an initial value problem where the unknown initial values $P(0)$ and $\phi(0)$ are adjusted iteratively until the end condition $\phi(a) = 0$ [eqn (4c)] is satisfied by the integral of eqns (3) and (4a, b). Repeated application of the shooting method yields a set of postbuckling solutions belonging to a one-parameter family. The delamination radius and the energy-release rate corresponding to each member solution can be straightforwardly evaluated. This determines the values of the energy-release rate in the entire course of growth of the delamination.

In terms of the nondimensionalized quantities

$$\xi = \frac{r}{a}, \qquad p = \frac{Pa^2}{D}, \qquad \Psi = [6(1 - \nu^2)]^{1/2} a^2 \frac{\phi}{h}, \tag{13a, b, c}$$

eqns (3)–(5) assume the following forms:

$$\xi^{-3}(\xi^3 p')' = \Psi^2, \qquad \xi^{-3}(\xi^3 \Psi')' = -p\Psi \tag{14}$$

$$p'(0) = \Psi'(0) = \Psi(1) = 0 \tag{15a, b, c}$$

$$p'(1) + (1 - \nu)p(1) = \frac{12(1 - \nu^2)a\delta}{h^2}, \qquad \delta = a\epsilon_0, \tag{16a, b}$$

where the primes now denote differentiation with respect to the new independent variable ξ. Substituting eqns (13), (15), (16) and (8a) into (12) delivers

$$\frac{G}{Eh\epsilon_0^2} = \frac{[p'(1)]^2(1 + \nu)/[2(1 - \nu)] + [\Psi'(1)]^2}{[p'(1) + (1 - \nu)p(1)]^2}. \tag{17}$$

For any given initial value $p(0)$, the initial value $\Psi(0)$ is searched for iteratively until the end condition $\Psi(1) = 0$ [eqn (15c)] is satisfied by the integral of eqns (14) and (15a, b). The resulting solution of eqns (14) and (15) determines the nondimensionalized delamination radius $\bar{a}$ and the nondimensionalized energy-release rate $\bar{G}$ via eqns (16) and (17), where

$$\bar{a} = [12(1 - \nu^2)\epsilon_0]^{1/2} \frac{a}{h} = [p'(1) + (1 - \nu)p(1)]^{1/2}, \qquad \bar{G} = \frac{G}{Eh\epsilon_0^2}.$$

For each new initial value $p(0)$, the shooting procedure is repeated, a new solution of eqns (14) and (15) is obtained and the corresponding values of $\bar{a}$ and $\bar{G}$ are computed. The integration of eqn (14) is executed by using the fourth-order Runge-Kutta formula[9] with a step size $\Delta\xi = \frac{1}{20}$. The formula for the beginning step is appropriately modified (by using l'Hôpital's rule) to deal with the apparent singularity of eqn (14) at $\xi = 0$. It is found that no more than 10 iterations in each shooting process suffice to yield a solution with an end value $|\Psi(1)| < 10^{-8}$. When the step size $\Delta\xi$ is reduced to $\frac{1}{30}$ and finally to $\frac{1}{50}$, the discrepancies found in the values of $\bar{a}$ and $\bar{G}$ are of the order of 10^{-4}. The initial and final values of a set of 22 solutions of eqns (14) and (15) are shown in Table 1, together with the corresponding values of $\bar{a}$ and $\bar{G}$ for a thin-film delamination with $\nu = 0.3$. Since the energy-release rate increases monotonically with the delamination radius, the growth of a circular thin-film delamination in a laminate under biaxial compressive strain, once started, is always catastrophic.

In Table 1, $p(0)$ decreases with increasing boundary compression $p(1)$ and assumes

Table 1. Nondimensionalized postbuckling solutions and the energy-release rate of a thin-film delamination

$\psi(0)$	$p(0)$	$p(1)$	$p'(1)$	$\psi'(1)$	$\bar{a}$	$\bar{G}$	$\bar{G}\bar{a}^4$
1.409	14.630	14.722	0.044	−1.138	3.217	0.0121	1.296
5.070	14.000	15.208	0.590	−4.221	3.352	0.1437	18.14
6.633	13.500	15.601	1.039	−5.658	3.458	0.2308	33.00
7.863	13.000	15.999	1.502	−6.873	3.564	0.3058	49.34
10.215	11.750	17.022	2.727	−9.501	3.827	0.4533	97.23
13.701	9.000	19.418	5.791	−14.666	4.403	0.6554	246.3
15.245	7.300	21.010	7.978	−17.850	4.763	0.7340	377.8
17.169	4.500	23.842	12.165	−23.426	5.372	0.8241	686.3
18.616	1.500	27.213	17.628	−30.156	6.056	0.8905	1198
19.152	0.000	29.046	20.815	−33.912	6.415	0.9169	1553
19.581	−1.500	30.991	24.358	−37.984	6.786	0.9401	1994
20.140	−4.400	35.103	32.382	−46.920	7.547	0.9789	3176
20.345	−6.500	38.407	39.343	−54.436	8.138	1.0033	4400
20.395	−9.000	42.750	49.162	−64.773	8.893	1.0296	6440
20.318	−10.680	45.948	56.863	−72.715	9.435	1.0459	8288
20.168	−12.300	49.267	65.263	−81.246	9.988	1.0609	10.56×10^3
19.947	−13.920	52.837	74.746	−90.741	10.570	1.0751	13.42×10^3
19.348	−17.000	60.380	96.266	−111.870	11.770	1.1005	21.12×10^3
18.568	−20.000	68.784	122.487	−137.027	13.063	1.1233	32.71×10^3
17.629	−23.000	78.334	154.989	−167.565	14.485	1.1444	50.38×10^3
16.570	−26.000	89.100	194.876	−204.330	16.039	1.1638	77.02×10^3
15.042	−30.000	105.379	261.288	−264.310	18.305	1.1870	133.3×10^3

negative values as $p(1)$ becomes sufficiently large. Hence, the membrane stress in a central portion of the delaminated layer changes from compression to tension as the edge load $p(1)$ increases with the growth of delamination. This result is in agreement with an earlier observation of Bodner[7], whose results were obtained by perturbation and asymptotic methods. The membrane tension in the central region stretches the layer, so that the curvature at the center [proportional to $\Psi(0)$] decreases. As delamination growth continues under increased laminate compression, the profile of the delamination flattens in a central region. Drastic changes in the altitude and the slope of the profile occur in a narrow region adjacent to the boundary of delamination. This yields an increased curvature of the profile and therefore an increased bending moment at the boundary, which in turn enhances the energy-release rate according to eqn (12).

If the constants h, a, E and ν and the fracture toughness of the material G^* are given, then by eliminating ϵ_0 from the defining expressions of $\bar{a}$ and $\bar{G}$, one finds that the following relation holds in all postbuckling states at incipient growth of delamination:

$$\bar{G}\bar{a}^4 = \frac{[12(1-\nu^2)a^2/h^2]^2 G^*}{Eh}.$$

Figure 2 shows the dependence of $\bar{G}$ and $\bar{G}\bar{a}^4$ upon $\bar{a}$ according to the solution of Table 1. This figure can be used to determine the critical laminate strain ϵ_0 at incipient growth of delamination. One first computes the value of $\bar{G}\bar{a}^4$ from the last equation. The $\bar{G}\bar{a}^4$ vs. $\bar{a}$ curve in the figure then yields the value of $\bar{a}$, which in turn delivers the critical laminate strain:

$$\epsilon_0 = \frac{(h\bar{a}/a)^2}{12(1-\nu^2)}.$$

5. GROWTH OF A CIRCULAR MIDPLANE DELAMINATION

The familiar problem of one-dimensional delamination growth in a symmetric, double-cantilever beam plate suggests a parallel problem concerning uniform-expansion growth of a circular midplane delamination at the center of a compressed circular plate.

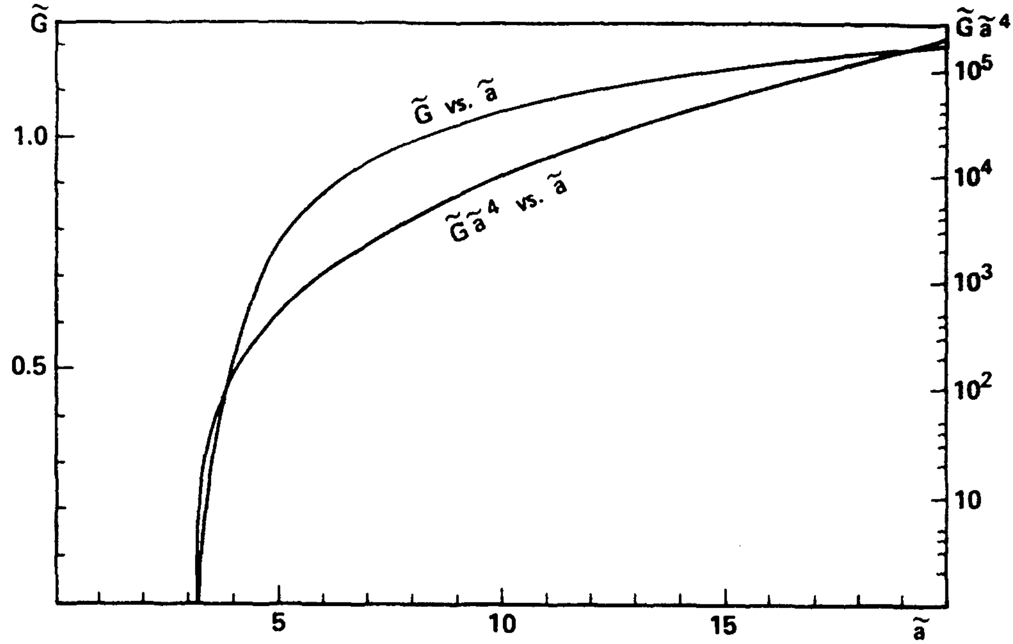

Fig. 2. $\tilde{G}$ and $\tilde{G}\tilde{a}^4$ vs. $\tilde{a}$ for a thin-film delamination.

Recently, Bottega and Maewal[4] gave an approximate analysis of the latter problem by means of the perturbation method. In a discussion of their work[5], I pointed out that the second-order perturbation method loses its validity in the range of loading corresponding to their delamination-growth path because the expansion parameter in the perturbation series becomes unduly large. In this section, the problem is reexamined by a refined analysis based on the one-parameter family of solutions of eqns (14) and (15).

For the present problem, eqns (13)–(15) and (16a) of the preceding section are applicable to the buckled layer on either side of the midplane. However, while δ, the inward radial displacement at the front of delamination, is determined by eqn (16b) in the thin-film problem because the laminate strain ϵ_0 is a fixed constant, in the present problem the laminate strain is not constant and δ is determined by the specified conditions at the exterior boundary of the laminate. The exterior boundary $\xi = R/a$ may be subjected to a force- or displacement-controlled compressive load. In the region $1 \leqq \xi \leqq R/a$, the transverse deflection of the laminate vanishes because the buckling deformation is symmetric with respect to the middle plane. Hence, the region is in a state of plane stress. An elementary elasticity solution delivers the radial force per unit arc length P_R and the radial displacement u_R at the exterior boundary in terms of the conditions at the interior boundary:

$$\bar{P}_R \equiv \frac{P_R R^2}{D} = \frac{[p(1) + p'(1)/2]}{a^2} - \frac{p'(1)}{2} \tag{18}$$

$$\bar{u}_R \equiv -\frac{EhRu_R}{D} = \frac{(1 - \nu)[p(1) + p'(1)/2]R^2}{a^2} + \frac{p'(1)(1 + \nu)}{2} \tag{19}$$

In obtaining these results, eqn (16a) was used to eliminate the radial displacement δ at the interior boundary.

Continuity of the radial stress at the front of a midplane delamination implies that P^* vanishes in eqn (11). Equation (11) actually gives only one-half of the M-integral associated with a midplane delamination since it was obtained by integrating over the circumferential boundary of one delaminated layer only. Hence, the total M-integral should be

$$M = \frac{24(1 - \nu^2)\pi(aM^*)^2}{Eh^3}.$$

This yields the energy-release rate of a midplane delamination:

$$G = \frac{M}{2\pi a^2} = \frac{(M^*)^2}{D}.$$

Using eqns (8a) and (13c), one obtains

$$\overline{G} \equiv \frac{[12(1 - \nu^2)R^2/h^2]^2 G}{2Eh} = \left[\frac{\Psi'(1)R^2}{a^2}\right]^2. \tag{20}$$

The first expression of eqn (20) was obtained in [4] by way of a different method.

While eqn (1b) gives the critical load of symmetric buckling of the two delaminated layers, the critical load of asymmetric buckling of the laminate as a whole (toward one side) is less than that of a perfect laminate of radius R and thickness $2h$. The critical buckling load of the perfect laminate $2P'_{cr}$ is determined by

$$R\left(\frac{2P'_{cr}}{8D}\right)^{1/2} = 3.8317.$$

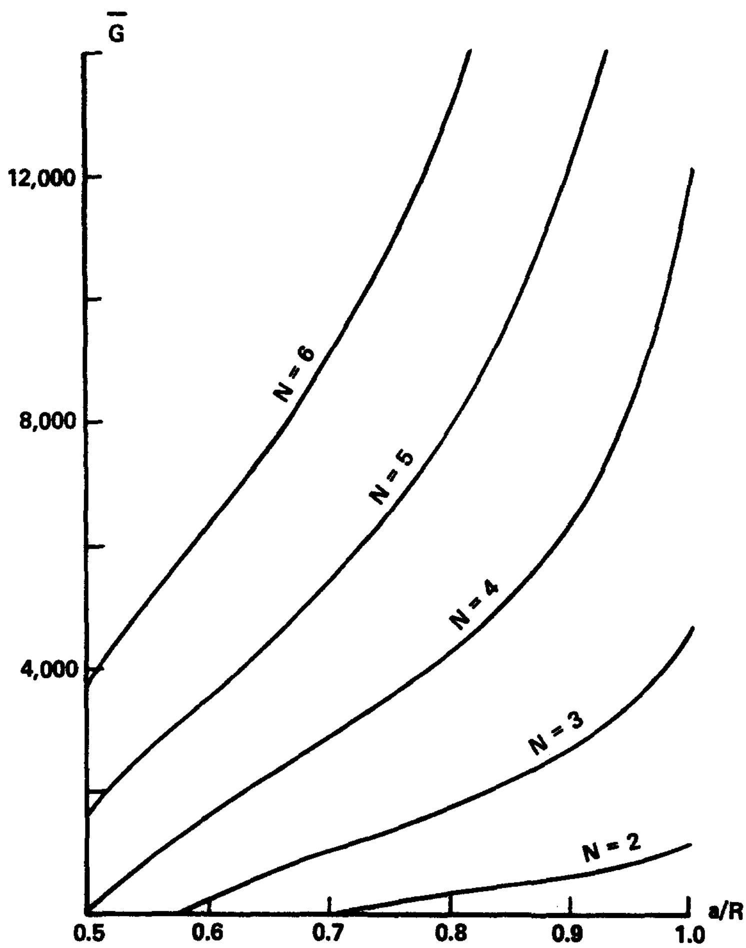

Fig. 3. $\overline{G}$ vs. a/R for a symmetric two-layer model under force-controlled loading ($N = \overline{P}_R/14.682$).

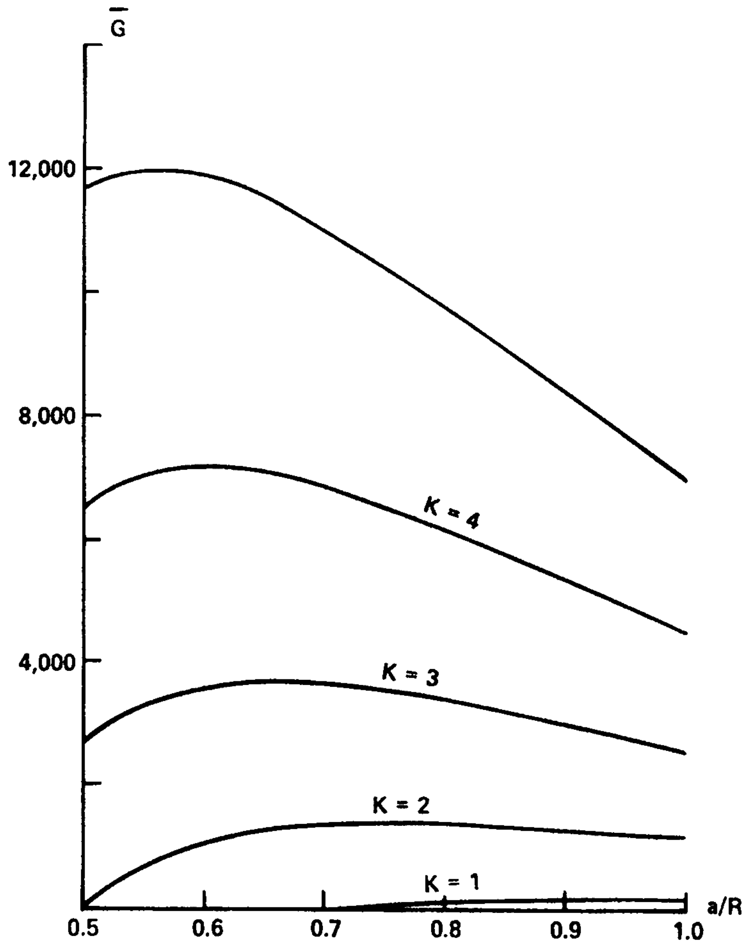

Fig. 4. $\overline{G}$ vs. a/R for a symmetric two-layer model under displacement-controlled loading $\{K \equiv \bar{u}_R/[2(1-\nu)14.682]\}$.

Comparing the last result with eqn (1b), one finds that symmetric buckling of the two delaminated layers will precede asymmetric buckling of the laminate only if

$$\frac{a}{R} > 0.5.$$

Henceforth, it is assumed that the radius of the existing midplane delamination exceeds $0.5R$.†

In the case of force-controlled loading, the left-hand side of eqn (18) is a specified constant. In a postbuckling state, eqns (18) and (20) yield the following expressions for the delamination radius and the energy-release rate:

$$\frac{a}{R} = \left[\frac{2p(1) + p'(1)}{2\overline{P}_R + p'(1)}\right]^{1/2}$$
$$\overline{G} = \left\{\frac{\Psi'(1)[2\overline{P}_R + p'(1)]}{2p(1) + p'(1)}\right\}^2. \qquad (21)$$

For a specified load $\overline{P}_R$, the data of the third to fifth columns of Table 1 generate pairs of values of a/R and $\overline{G}$ via eqn (21). The dependence of $\overline{G}$ upon a/R is shown in Fig. 3 for several values of $\overline{P}_R$ and for $\nu = 0.3$.

† Symmetrical buckling can be ensured by providing movable transverse supports in the region outside the delamination.

Similarly, in the case of displacement-controlled loading, eqns (19) and (20) deliver

$$\frac{a}{R} = \left\{\frac{(1 - \nu)[2p(1) + p'(1)]}{2\overline{u}_R - (1 + \nu)p'(1)}\right\}^{1/2}$$
$$\overline{G} = \left\{\frac{2\overline{u}_R - (1 + \nu)p'(1)}{(1 - \nu)[2p(1) + p'(1)]}\Psi'(1)\right\}^2. \tag{22}$$

The dependence of $\overline{G}$ upon a/R is shown in Fig. 4 for $\nu = 0.3$ and several values of $\overline{u}_R$.

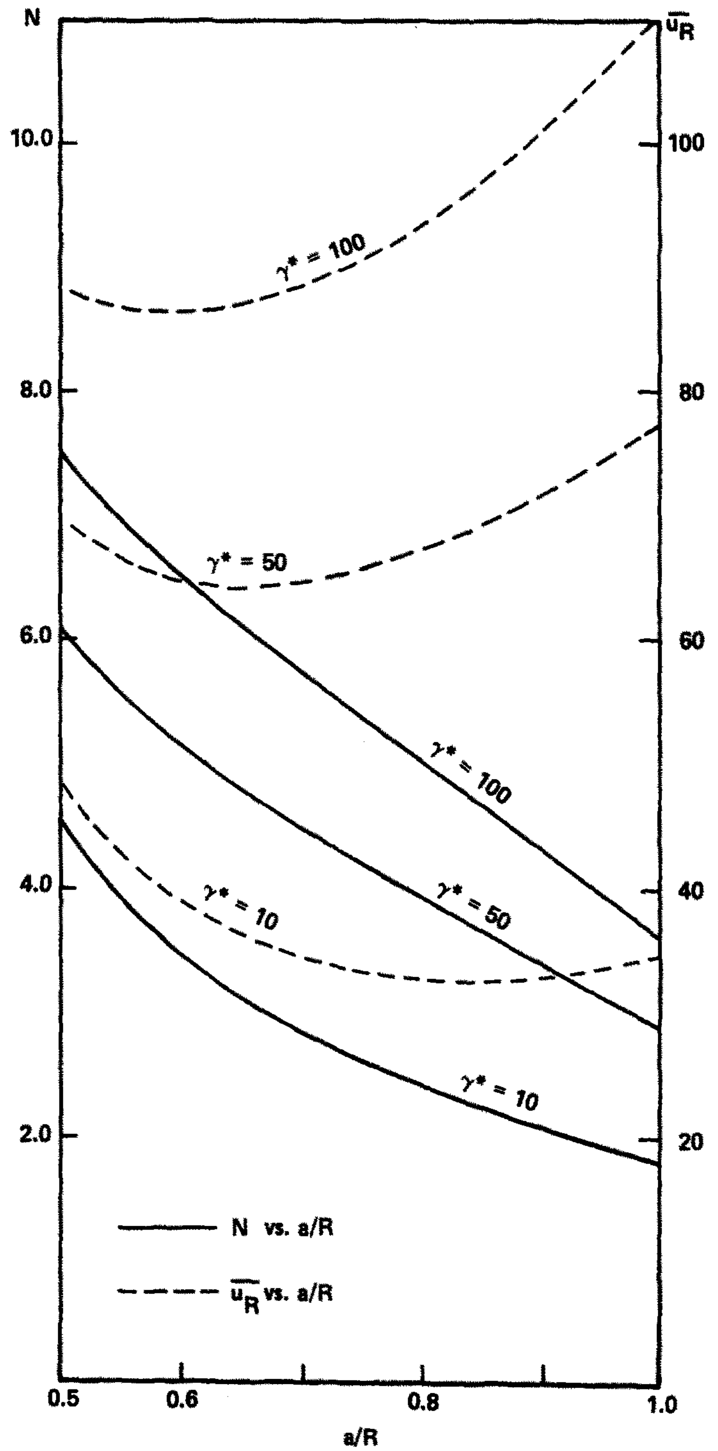

Fig. 5. N and $\overline{u}_R$ vs. a/R for constant values of $\overline{G}\{\gamma^* = \overline{G}/[6(1 - \nu^2)14.682]\}$.

Since the energy-release rate is a monotonically increasing function of the delamination radius under force-controlled loading (Fig. 3), the growth of delamination is always catastrophic once it has started. This conclusion directly contradicts the result of [4], and demonstrates the importance of a refined postbuckling solution in the analysis of delamination growth.

Another way to examine the stability property of delamination growth is by considering the boundary forces or boundary displacements required to maintain a constant value of the energy-release rate (the fracture toughness of the material) in the successive stages of delamination growth. This consideration defines a "delamination growth path" as introduced by Bottega and Maewal[4]. Along such a path ($\overline{G}$ = constant), the change of the boundary force and the boundary displacement with the increase of delamination radius may also be determined from eqns (18)–(20) on the basis of the data in Table 1. The dependence of $N \equiv \overline{P}_R/14.682$ and $\overline{u}_R$ upon a/R is given in Fig. 5 for $\nu = 0.3$ and for three values of $\overline{G}$ corresponding to†

$$\gamma^* \equiv \frac{\overline{G}}{6\,(1 - \nu^2)14.682} = 10,\ 50,\ 100.$$

It is seen that the boundary force required to maintain a constant level of energy-release rate decreases monotonically as delamination continues.

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† Figure 4 in [4] shows the relation between N and a/R for $\gamma^* = 50$ according to the results of the perturbation analysis. As a/R increases from 0.5 to 1.0, the perturbation analysis of [4] predicts that N increases from 10.0 to 17.5, whereas Fig. 5 of the present paper shows that N decreases from 6.1 to 2.9.